MODEL PREDICTIVE SLIDING MODE CONTROL — FOR CONSTRAINT SATISFACTION AND ROBUSTNESS

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ABSTRACT

A novel combination of model predictive control (MPC) and sliding mode control (SMC) is presented in this paper. The motivation is to inherit the ability to explicitly deal with state and input constraints from MPC, and the good robustness property from SMC. The design of the finite-time optimal control problem and the conditions for the persistent feasibility and the closed-loop stability are discussed. Simulation results are shown to demonstrate the nominal and robust performance of the proposed control algorithm.

INTRODUCTION

The idea of the model predictive control (MPC) is to predict the system evolution in future time instances by using a model of the system [1–3]. At each time step, a certain objective function is optimized over a sequence of future control inputs subject to operating constraints. The model predictive control provides an approximation of an infinite-horizon optimal control by using the receding horizon technique. The system constraints in the optimization problem include modeled system dynamics, actuator saturations, system state limits due to regulations and safety considerations in practice. It is the capability of dealing with constraints explicitly that makes MPC distinct from other control techniques.

When it comes to handling the system model uncertainties or external unexpected disturbances, the analysis of the closed-loop feasibility, stability, and robustness of the MPC becomes difficult because of the computational complexity, design of objective function, and terminal constraints, etc. Robust MPC formulations have been extensively studied in [3–5]. However, the computational complexity is normally the major bottleneck of practical applications of these developed robust MPC methods.

On the other hand, sliding mode control (SMC) has been developed to deal with uncertainties [6–8]. The idea is to allow the transformation of a controller design problem for a general n-th order system to a simple first-order stabilization problem, i.e., stabilizing the dynamics associated with the switching function. Then for the equivalent first-order system, the intuitive feedback control strategy can be applied, "if the error is negative, push hard enough in the positive direction and conversely" [9]. Good performance can be guaranteed even in the presence of parametric uncertainties. The discrete-time counterpart has also been well studied in [10–12], which is important for digital control systems with relatively slow sampling rate.

The standard SMC, however, fails to address the practical issue of the hard constraints often imposed on the system state and/or input. Some past works [13–15] were conducted to investigate the constrained SMC for the single-input linear time invariant system. Although these methods can deal with the state constraint [13, 14] or the output constraint [15], they have not considered the input constraint together with other constraints, which can normally be dealt with by MPC.

Therefore, it is a natural step to investigate the possibility of combining the two control techniques (i.e., MPC and SMC), to take advantages of coping with system constraints of MPC and robustness property of SMC. In [16, 17], the first attempt has been made to analyze the stability of such a closed-loop system. The proposed methodology in these past works was inspiring but inadequate. For example, the persistent feasibility condition was not investigated. Also, the closed-loop stability was discussed only with perfect model assumption. In this paper, we will further generalize their work by studying the persistent feasibility, computing the zero-th step feasible set, and proving the closed-loop stability for both a perfect nominal model and an uncertain model.

This paper is organized as follows. Short reviews of the basic discrete-time SMC and MPC are first provided. The novel model predictive sliding mode control is then proposed, along with the sliding mode conditions for the persistent feasibility and the closed-loop stability. For robust performance in the presence of uncertainty, we then discuss a min-max formulation. Finally, a simulation study is presented to demonstrate both the nominal and the robust performance.

PRELIMINARIES

Discrete-Time Sliding Mode Control (DT-SMC)

Consider a discrete-time single-input linear time-invariant (DT-SI-LTI) system in the normal form with no parametric uncertainty, disturbance, state or input constraints, i.e.

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^n$. Sliding mode control design consists of two steps: (1) design a stable sliding manifold $s(x) = c^T x = 0$ (with $c^T B \neq 0$), so that $x(t)$ approaches to the origin when $s = 0$, and (2) design a reaching law and the corresponding control input so that $s(x)$ is attracted to 0.

For the design of the stable sliding manifold and the stability analysis during the sliding mode, without loss of generality, we assume that the original system is controllable, and we transform it to the normal form, i.e.

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0, \dots, 1 \end{bmatrix}^T \\ x &= \begin{bmatrix} x_1, x_2 \end{bmatrix}^T, \quad c = \begin{bmatrix} c_1 \\ 1 \end{bmatrix} \end{aligned} \tag{2}$$

where A_{22} and x_2 are scalars. During the sliding mode $s(x) = c_1^T x_1 + x_2 = 0$, we can express the dynamics of x_1 as

$$x_1(t+1) = (A_{11} - A_{12}c_1^T)x_1(t) \tag{3}$$

Controllability of the original system implies that $\{A_{11}, A_{12}\}$ pair is also controllable [18]. Hence the eigenvalues of $A_{11} - A_{12}c_1^T$ can be arbitrarily assigned by the choice of c_1. For stability, they must be inside the unit circle. In order to enforce the state to remain on the sliding manifold (i.e., $s(t+1) = s(t) = 0$), the equivalent control is given by

$$u_{eq}(x(t)) = -(c^T B)^{-1} c^T Ax(t) \tag{4}$$

The dynamics corresponding to this ideal sliding mode becomes

$$x(t+1) = [I_{n\times n} - B(c^T B)^{-1} c^T]Ax(t) \tag{5}$$

where $I_{n\times n}$ is the identity matrix in $\mathbb{R}^{n\times n}$. Through simple algebraic manipulations, one can show that $[I_{n\times n} - B(c^T B)^{-1} c^T]A$ in Eqn. (5) always has an eigenvalue at the origin plus the same eigenvalues of $A_{11} - A_{12}c_1^T$ in Eqn. (3), if the system is in the normal form.

When the sliding mode controller is designed in discrete-time, or the continuous-time design is implemented digitally, or the discrete-time model (1) is a sampled model representation of an actual continuous-time plant, the state variable may not be exactly on the sliding manifold, instead it zigzags along $s = 0$ within a sliding mode band $|s| \leq \varepsilon$ [11]. $x_1(t)$ has an additional forced response term. The stability may be inferred from

$$\begin{aligned} x_1(t+1) &= (A_{11} - A_{12}c_1^T)x_1(t) + A_{12}s(t), \quad \forall |s(t)| \leq \varepsilon \\ x_2(t) &= s(t) - c_1^T x_1(t) \end{aligned} \tag{6}$$

which may not cause problem since both the eigenvalues and the sliding mode band width can be designed to be small.

Gao *et. al.* proposed the following quasi-sliding mode reaching law [11]

$$\begin{aligned} s(t+1) - s(t) &= -\sigma T s(t) - \mu T \operatorname{sgn}(s(t)) \\ \sigma > 0,\ \mu &> 0,\ 1 - \sigma T > 0 \end{aligned} \tag{7}$$

where $T > 0$ is the sampling period, μ and σ are the design parameters, which guarantee the following desired attributes: (1) starting from any initial state, the state trajectory monotonically move toward the sliding manifold; (2) once the trajectory has crossed the manifold the first time, it zigzags along and crosses the manifold in every successive step; (3) the trajectory is confined in the sliding mode band (SMB) stated as

$$SMB = \{x \in \mathbb{R}^n \big| \, |s(x)| < \frac{\mu T}{1 - \sigma T}\} \tag{8}$$

The control that enforces the reaching law is given by

$$u(t) = -(c^T B)^{-1} \left[c^T Ax(t) - c^T x(t) + \sigma T c^T x(t) + \mu T \operatorname{sgn}(c^T x(t))\right] \tag{9}$$

Therefore the stability of the closed-loop system can be achieved by choosing c_1 to place the eigenvalues in Eqn. (6) inside the unit circle and choosing μ, σ to make the forced response term negligible.

Discrete-Time Model Predictive Control (DT-MPC)

Consider the same DT-SI-LTI system with state and input constraints as

$$x(t) \in \mathscr{X},\ u(t) \in \mathscr{U},\ \forall t \geq 0 \tag{10}$$

Assume the constraints are in the form of convex polyhedra. Namely, $\mathscr{X} = \{x \in \mathbb{R}^n \big| A_x x \leq b_x\}$, where $A_x x \leq b_x$ is the usual notation for the intersection of m_x closed halfspaces, each represented by $a_{xi} x \leq b_{xi}, i = 1, \ldots, m_x$. Similarly, $\mathscr{U} = \{u \in \mathbb{R} \big| A_u u \leq b_u\}$. We assume that $\mathscr{X}, \mathscr{U}$ contain the origin in their interior and are closed.

The model predictive control provides an approximation of an infinite-horizon optimal control by using the receding horizon technique. Suppose at time step t, the state variable $x(t)$ is measured, the following constrained finite time optimal control (CFTOC) problem is solved

$$\begin{aligned} J_0^*(x(t)) &= \min_{U_{0\to N-1}} J_0(x(t), U_{0\to N-1}) \\ &\triangleq \min_{U_{0\to N-1}} p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) \\ \text{subj. to} \quad & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & x_k \in \mathscr{X},\ u_k \in \mathscr{U},\ k = 0, \ldots, N-1 \\ & x_N \in \mathscr{X}_f,\ x_0 = x(t) \end{aligned} \tag{11}$$

where the terminal constraint $\mathscr{X}_f$ is also a convex polyhedron, $p(x_N)$ and $q(x_k, u_k)$ are the terminal cost and the stage cost respectively, N is the prediction horizon, $(\bullet)_k$ represents the *predicted* state or input at time $t+k$ based on the knowledge of the state at time t, $U_{0\to N-1} = [u_0^T, \ldots, u_{N-1}^T]^T$ contains the decision variables, that is, the N control inputs from time t to $t+N-1$. After solving the above optimization problem, we apply only the first optimal control $u(t) = u_0^*(x(t))$ to the real system. We solve the CFTOC problem again at the next time instance with $x_0 = x(t+1)$. The closed-loop system can be expressed as

$$x(t+1) = Ax(t) + Bu_0^*(x(t)) = f_{cl}(x(t)), t \geq 0 \tag{12}$$

Two desired properties associated with MPC are of great interest, which are the persistent feasibility and the closed-loop stability. Without them, MPC may lead us into a situation where the aforementioned CFTOC problem becomes infeasible after a few steps of the system evolution, or the generated control inputs may not lead to an asymptotically converging closed-loop state trajectory.

Lemma 1. **Persistent feasibility** *[1]: Consider the model predictive control law described by Eqns.(11)-(12) with the prediction horizon $N \geq 1$, if $\mathscr{X}_f$ is a control invariant set for the system (Eqns.(1) and (10)), then the MPC is persistently feasible.*

Control invariance means that once the system state enters the set, there exists a control input that can confine the state in this set hereafter. The proof of the lemma is given in [1].

NOMINAL MODEL PREDICTIVE SLIDING MODE CONTROL (MP-SMC)

Sliding mode control inspires us that, instead of directly stabilize the original n-th order system, it is much easier to design a controller that can stabilize the transformed first-order system, namely the dynamics of the switching function $s = c^T x$. The complexity of the design is significantly reduced especially when there are uncertainties, i.e., parametric variations and unexpected disturbances, present in the system.

Model predictive control is a control methodology that can explicitly deal with constraints. It is naturally motivated to investigate whether we can particularly design a CFTOC problem so that the designed stable switching function $s = c^T x$ converges to 0 after we keep applying the first optimal control from the CFTOC optimization with receding horizon to the system. Pedagogically, it is easier to first discuss the persistent feasibility and the closed-loop stability in the absence of uncertainties.

In light of the persistent feasibility lemma, a control invariant terminal set must be designed. The following condition relates the input power with the terminal constraint design.

Control Invariance Condition

For $\varepsilon > 0$ such that the system represented by Eqn. (6) is stable, we design the terminal constraint $\mathscr{X}_f \subseteq \mathscr{X}$ such that $\forall x \in \mathscr{X}_f \subseteq \{x \in \mathscr{X} \big| |s(x)| \leq \varepsilon\}$,

$$u_{sm}(x) = -(c^T B)^{-1} c^T A x \in \mathscr{U} = \{u \in \mathbb{R} \big| A_u u \leq b_u\} \tag{13}$$

where the sliding mode control law comes from Eqn. (4). The design of $\mathscr{X}_f$ can be easily done by first assuming the system is in the normal form. It can be readily seen that

$$\begin{aligned} u_{sm}(x) &= [F_1\ F_2] \begin{bmatrix} x_1 \\ s \end{bmatrix} \\ F_1 &= -(c^T B)^{-1} [c_1^T (A_{11} - A_{12} c_1^T) + A_{21} - A_{22} c_1^T] \\ F_2 &= -(c^T B)^{-1} [c_1^T A_{12} + A_{22}] \end{aligned} \tag{14}$$

It is desired to have the following inequalities

$$\begin{aligned} & A_u F_1 x_1 \leq b_u - A_u F_2 s, \quad \forall |s| \leq \varepsilon \\ \Rightarrow \quad & A_u F_1 x_1 \leq \min_{|s| \leq \varepsilon} \{b_u - A_u F_2 s\} \end{aligned} \tag{15}$$

where we use the shorthand notation $\min_s [a] = [\ldots, \min_s a_i, \ldots]^T$ for vectors. Writing the above inequality in x_1 as $A_r x_1 \leq b_r$, we can obtain $\mathscr{X}_f = \{x \in \mathscr{X} \,|\, A_f x \leq b_f\}$ as

$$A_f = \begin{bmatrix} c_1^T & 1 \\ -c_1^T & -1 \\ A_r & 0 \\ \hline \multicolumn{2}{c}{A_x} \end{bmatrix}, \quad b_f = \begin{bmatrix} \varepsilon \\ \varepsilon \\ b_r \\ b_x \end{bmatrix} \tag{16}$$

The assumption that $\mathscr{X}$ contains the origin and is closed implies that $\mathscr{X}_f$ has the same properties.

Although, with $\mathscr{X}_f$ designed above, the control is guaranteed to have the power to steer the state to be exactly on the sliding manifold at the next time step, i.e. $c^T(x(t+1)) = 0$ if $x(t) \in \mathscr{X}_f$. It is, however, not necessarily the case that $x(t+1) \in \mathscr{X}_f$. This might be counterintuitive since in a stable sliding mode band the state tends to move toward the origin, instead of moving away from it. In order to achieve the control invariance property of $\mathscr{X}_f$, the following geometric condition must also be satisfied

$$\mathscr{X}_f \subseteq \tilde{\mathscr{X}}_f \tag{17}$$

where

$$\tilde{\mathscr{X}}_f \triangleq \{x \in \mathscr{X} \,|\, A_f(I_{n\times n} - B(c^T B)^{-1} c^T)Ax \leq b_f\} \tag{18}$$

MP-SMC Formulation

The proposed MP-SMC solves the following CFTOC problem with receding horizon

$$\begin{aligned} J_0^*(x(t)) &= \min_{U_{0\to N-1}} J_0(x(t), U_{0\to N-1}) \\ &\triangleq \min_{U_{0\to N-1}} \sum_{k=0}^{N} |c^T x_k| \\ \text{subj. to} \quad & x_{k+1} = Ax_k + Bu_k,\ k = 0,\dots,N-1 \\ & x_k \in \mathscr{X},\ u_k \in \mathscr{U},\ k = 0,\dots,N-1 \\ & x_0 = x(t),\ x_N \in \mathscr{X}_f \end{aligned} \tag{19}$$

where $s(t) = c^T x(t)$ is the designed stable switching function with sliding mode band width $\varepsilon > 0$. $\mathscr{X}_f$ is obtained following the design procedure discussed in the control invariance condition. Since it is desirable to make the state converge to the sliding manifold $s = 0$, we penalize the deviation of the switching function from 0 for the entire prediction horizon.

Remark 1. *The CFTOC problem Eqn.* (19) *can be written in the form of multi-parametric linear programming. The feasible set (the zero-th step feasible set), denoted as $\mathscr{X}_0$, is a polyhedron. The optimal objective function $J_0^*(x) : \mathscr{X}_0 \to \mathbb{R}$ is continuous, convex and piecewise affine over $\mathscr{X}_0$. The optimal solution $U^*_{0\to N-1}$ is continuous piecewise affine over $\mathscr{X}_0$ [1].*

For the persistent feasibility and the closed-loop stability, we have the following theorem:

Theorem 1. Closed-loop stability: *For the CFTOC problem in Eqn.* (19) *with the assumptions* (16)-(17) *described above, $\mathscr{X}_f$ is control invariant. The problem is persistently feasible for any state in $\mathscr{X}_0$. The switching function $s(t)$ of the closed-loop system converges to 0 as $t \to \infty$. The stability of the manifold guarantees that the closed-loop system state asymptotically converges to the origin, i.e.* $\lim_{t\to\infty} x(t) = \mathbf{0}$. *The domain of attraction is $\mathscr{X}_0$.*

Proof. The persistent feasibility can be concluded directly from the control invariance condition and the persistent feasibility lemma. The closed-loop stability is proved by establishing that $J_0^*(\bullet)$ in Eqn. (19) is a Lyapunov function for the closed-loop system. We first note that $J_0^* : \mathbb{R}^n \to \mathbb{R}$ is "boat-shaped", i.e.

$$\begin{aligned} &\bullet\ J_0^*(x) \geq 0, \quad \forall x \in \mathscr{X}_0 \\ &\bullet\ J_0^*(x) = 0, \quad \text{only when } c^T x = 0 \\ &\bullet\ J_0^*(x) \text{ is a convex function of } x,\ \forall x \in \mathscr{X}_0 \end{aligned} \tag{20}$$

The change of $J_0^*(x(t))$ along the state trajectory is then investigated. Consider $x(t) \in \mathscr{X}_0$, suppose Eqn. (19) has the minimizer $U_0^* = \{u_0^*, \dots, u_{N-1}^*\}$ and the corresponding optimal state trajectory $\boldsymbol{x}_0 = \{x_0, \dots, x_N\}$. We apply u_0^* to the system and obtain $x(t+1) = Ax(t) + Bu_0^* = x_1$ since perfect model is assumed. At time $t+1$, Eqn. (19) is solved again for $x_0 = x(t+1)$. An upper bound of $J_0^*(x(t+1))$ can be constructed by finding a feasible control sequence, $\tilde{U}_0 = \{u_1^*, \dots, u_{N-1}^*, u_{sm}(x_N)\}$. Since $x_N \in \mathscr{X}_f$, by the control invariance condition, $u_{sm}(x_N) \in \mathscr{U}$, $x_{N+1} \in \mathscr{X}_f$, and $c^T x_{N+1} = 0$, i.e., the input constraint and the terminal set constraint are satisfied by the proposed control sequence. The objective function corresponding to $\tilde{U}_0$ is $J_0(x(t+1), \tilde{U}_0)$, which provides an upper bound for the optimal objective function value, i.e.

$$\begin{aligned} J_0^*(x(t+1)) \leq J_0(x(t+1), \tilde{U}_0) = J_0^*(x(t)) - |c^T x(t)| + |c^T x_{N+1}| \\ \Rightarrow \quad J_0^*(x(t+1)) - J_0^*(x(t)) \leq -|c^T x(t)| \end{aligned} \tag{21}$$

Therefore, we can regard $J_0^*(\bullet)$ as a Lyapunov function of $s(t)$. Using Lyapunov direct theorem, $s(t)$ of the closed-loop system converges to 0 as $t \to \infty$. Therefore, the closed-loop state x converges to the origin along the designed stable sliding manifold.

ROBUST MODEL PREDICTIVE SLIDING MODE CONTROL

Uncertain Model

The MP-SMC designed in the previous section is based on the nominal model without parametric uncertainty or external disturbances. In practice, the nominal model is often subject to these model uncertainties. We now consider an uncertain single input LTI system model in the following form

$$x(t+1) = (A + \Delta A)x(t) + Bu(t) + f(t) \tag{22}$$

where ΔA and $f(t)$ have the compatible dimension. We assume the matching conditions: $\Delta A = B\bar{A}$ and $f = B\bar{f}$ ($\bar{A} \in \mathbb{R}^{1\times n}, \bar{f} \in \mathbb{R}$). Denote the lumped uncertainty as $d(t) = \bar{A}x(t) + \bar{f}$, and assume it is known a-*priori* to be within some bound, i.e.

$$\underline{d} \leq (c^T B)d(t) \leq \overline{d} \tag{23}$$

The following notation is introduced

$$d_{av} = \frac{\overline{d}+\underline{d}}{2}, \quad \delta d = \frac{\overline{d}-\underline{d}}{2} \\ \mathscr{D} = [d_{av} - \delta d, d_{av} + \delta d] \tag{24}$$

Eqn. (22) can be rewritten as

$$x(t+1) = Ax(t) + B(u(t) + d(t)) \tag{25}$$

Remark 2. *Note that we are only studying the case where the model uncertainty is matched into the input channel and bounded. If the matching conditions and uncertainty bound condition are not valid, another analysis and design will need to be resorted to for the mismatched uncertainty.*

Similar to the sliding mode control, our proposed algorithm regulates the switching function $s(x)$ near 0, instead of regulating the state variable x directly.

Lemma 2. Robust sliding mode condition*: It is first noted that the robust sliding mode band $RSMB = \{x \in \mathbb{R}^n \big| \, |s(x)| \leq \delta d\}$ is robust control invariant to the following control*

$$u_{rsm}(x) = -(c^T B)^{-1}(c^T Ax(t) + d_{av}) \tag{26}$$

With this control and regardless of the actual disturbance, the state at the next time step is guaranteed to stay in RSMB, i.e.

$$s(t+1) = -d_{av} + (c^T B)d(t) \in [-\delta d, \delta d], \; \forall c^T Bd(t) \in \mathscr{D} \tag{27}$$

The procedure is similar to the nominal case to design a polytope $\mathscr{X}_{f,robust} \subseteq \mathscr{X}$ such that $u_{rsm}(x) \in \mathscr{U}, \forall x \in \mathscr{X}_{f,robust} \subseteq \{x \in \mathscr{X} \big| \, |s(x)| < \delta d\}$.

To achieve the robust control invariance property of $\mathscr{X}_{f,robust}$, a more restricted condition involving the system uncertainty is derived as follows

$$A_f(I_{n\times n} - B(c^T B)^{-1}c^T)Ax + A_f B(c^T B)^{-1}d' \leq b_f, \\ \forall x \in \mathscr{X}_f, \forall d' \in [-\delta d, \delta d] \tag{28}$$

Geometrically, if we denote Eqn. (28) as the polytope $\tilde{\mathscr{X}}_{f,robust}$ in x, the following condition needs to be verified,

$$\mathscr{X}_{f,robust} \subseteq \tilde{\mathscr{X}}_{f,robust} \tag{29}$$

Combining these conditions, $\forall x \in \mathscr{X}_{f,robust}$, there exists $u_{rsm}(x) = -(c^T B)^{-1}(c^T Ax(t) + d_{av})$, which satisfies the input constraint and can keep the state within $\mathscr{X}_{f,robust}$. Essentially, this condition drives the state to move toward the origin when it slides in the robust sliding mode band.

Min-Max CFTOC with Closed-Loop Prediction

For robust performance in the presence of uncertainty, we want to optimize the controller performance for the worst case, i.e., to minimize the objective function subject to the worst possible uncertainty. Therefore we pose the CFTOC as the following min-max optimization problem which needs to be solved recursively backwards from the terminal step (referred to as the closed-loop prediction).

$$J_j^*(x_j) = \min_{u_j} J_j(x_j, u_j) \\ \text{subj. to} \begin{cases} x_j \in \mathscr{X}, \quad u_j \in \mathscr{U} \\ Ax_j + B(u_j + d_j) \in \mathscr{X}_{j+1}, \quad \forall c^T Bd_j \in \mathscr{D} \end{cases} \tag{30}$$

where

$$J_j(x_j, u_j) \triangleq \max_{d_j} |c^T x_j| + J_{j+1}^*(x_j, u_j, d_j) \tag{31}$$

The j-th step feasible set is also computed recursively,

$$\mathscr{X}_j = \{x \in \mathscr{X} \mid \exists u \in \mathscr{U} \text{ such that} \\ Ax_j + B(u_j + d_j) \in \mathscr{X}_{j+1}, \quad \forall c^T Bd_j \in \mathscr{D}\} \tag{32}$$

for $j = 0, \ldots, N-1$ and with boundary conditions

$$J_N^*(x_N) = |c^T x_N| \\ \mathscr{X}_N = \mathscr{X}_{f,robust} \tag{33}$$

where $\mathscr{X}_{f,robust}$ satisfies the robust sliding mode condition. With the help from the one-dimensional switching function $s(x)$, it is not hard to quantify the "worst case" disturbance, which is the one pushing s away from 0. The above min-max formulation is also a multi-parametric linear programming. The properties discussed in Remark 1 hold.

Theorem 2. Robust closed-loop stability*: For the min-max formulation described in Eqns. (30)-(33), the problem is persistently feasible for any state in $\mathscr{X}_0$. The switching function $s(t) = c^T x(t)$ of the closed-loop system converges to $\{s \big| |s| \leq \delta d\}$ as $t \to \infty$. Furthermore, x stays close to the origin since the sliding manifold is designed stable.*

Proof. The persistent feasibility can be concluded from the robust control invariance of the terminal set constraint. Considering $x(t) \in \mathscr{X}_0$, the proposed CFTOC problem described in Eqns. (30)-(33) is solved recursively to provide the optimal control sequence $U_0^* = \{u_0^*, \ldots, u_{N-1}^*\}$. $x_{N|t}$, which denotes the N-th step forward predicted state based on the knowledge at time t, is in $\mathscr{X}_{f,robust}$ regardless of the predicted disturbances. Let $d_{\bullet|t} = \{d_0, d_1, \ldots, d_{N-1}\}$ denote the worst-case disturbances.

Consider the objective function value $J_0'(x(t))$ corresponding the optimal control U_0^* and a slightly modified disturbance sequence $\{d(t), d_1, \ldots, d_{N-1}\}$, where $d(t)$ is the actual disturbance acting to the plant at time t. By the min-max nature, we have $J_0'(x(t)) \leq J_0^*(x(t))$. It certainly follows that

$$J_0'(x(t)) + \delta d \leq J_0^*(x(t)) + \delta d \tag{34}$$

We apply the first optimal control u_0^* to the plant and the plant is also subject to the disturbance $d(t)$. The state at the next time is $x(t+1) = Ax(t) + B(u_0^* + d(t))$. The following non-optimal control $\tilde{U}_0^* = \{u_1^*, \ldots, u_{N-1}^*, u_{rsm}(x_{N|t+1})\}$ must be feasible since

$$\begin{aligned} x_{N-1|t+1} &\in \mathscr{X}_{f,robust} \\ u_{rsm}(x_{N-1|t+1}) &\in \mathscr{U} \\ x_{N|t+1} = Ax_{N-1|t+1} + Bu_{rsm}(x_{N-1|t+1}) &\in \mathscr{X}_{f,robust} \end{aligned} \tag{35}$$

The corresponding objective function value $J_0^o(x(t+1))$ is greater than the optimal value, i.e., $J_0^*(x(t+1)) \leq J_0^o(x(t+1))$. It follows that

$$|c^T x(t)| + J_0^*(x(t+1)) \leq |c^T x(t)| + J_0^o(x(t+1)) \tag{36}$$

Note that by construction, the left hand side of Eqn. (34) and the right hand side of Eqn. (36) should have the same value because

(i) they have the same initial conditions $x(t)$ and $d(t)$
(ii) u_j^*'s are used for $j = 0, 1, \ldots, N-1$
(iii) $\max_{d_{N-1|t+1}} |c^T x_{N|t+1}| = \delta d$ since $x_{N-1|t+1} \in \mathscr{X}_{f,robust}$

Therefore, from Eqns. (34) and (36), we have

$$J_0^*(x(t+1)) - J_0^*(x(t)) \leq \delta d - |c^T x(t)| \tag{37}$$

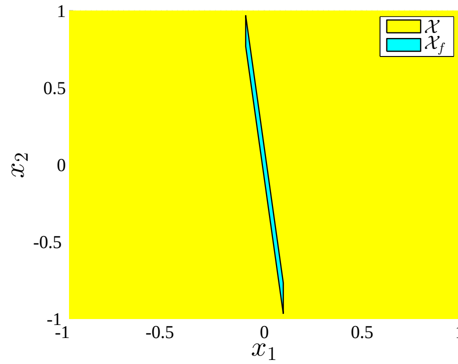
Note that, $J_0^* : \mathbb{R}^n \to \mathbb{R}$ is still "boat-shaped" since

- $J_0^*(x)$ achieves its minimum J^* only when $|c^T x| = 0$
- $J_0^*(x) \geq J^*, \quad \forall x \in \mathscr{X}_0$
- $J_0^*(x)$ is a convex function of x, $\forall x \in \mathscr{X}_0$

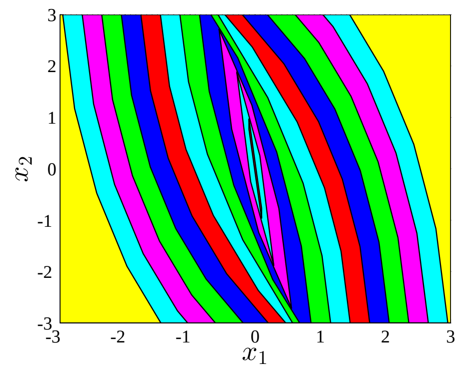
Furthermore, as seen from (37), J_0^* stops decreasing only when $x \in \{x \in \mathscr{X} \,\big|\, |c^T x(t)| \leq \delta d\}$. Therefore, as $t \to \infty$, the state is attracted to the robust sliding mode band and stays near the origin since the sliding manifold is stable.

SIMULATION RESULTS

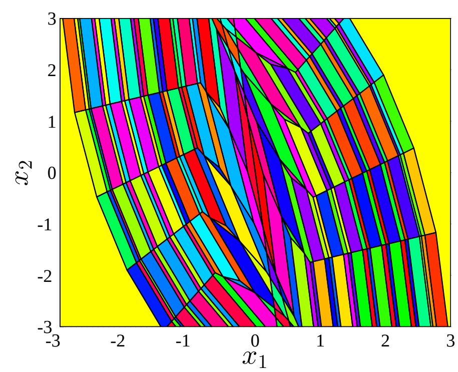
In this section, we demonstrate the performance of the proposed model predictive sliding mode control using numerical simulations.



(a) The design of $\mathscr{X}_f$ enforcing the sliding condition



(b) The j-th step feasible set (the set expands with decreasing j)

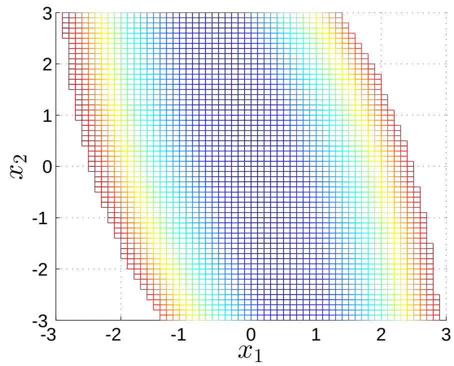


(c) The critical regions of the 0-th step feasible set

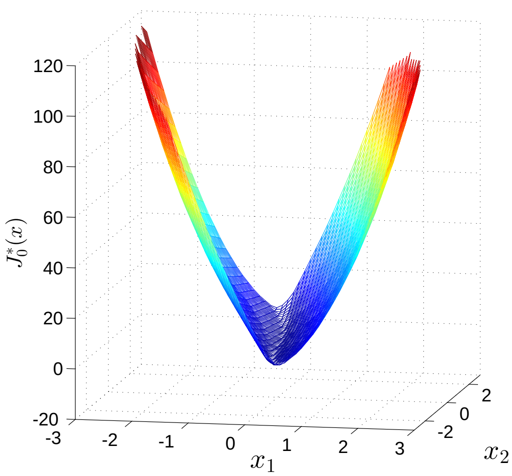
Figure 1. Nominal performance: the feasible sets

Nominal Performance

Consider a DT-SI-LTI system in the normal form. For nominal performance validation, perfect model is assumed for simulation without model uncertainty. We use a second-order system described below to demonstrate the performance in order to clearly visualize the constraints and the optimal objective func-

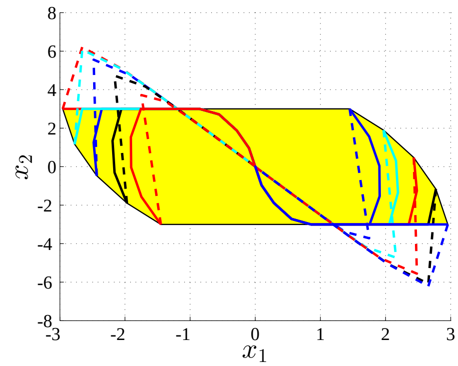


(a)

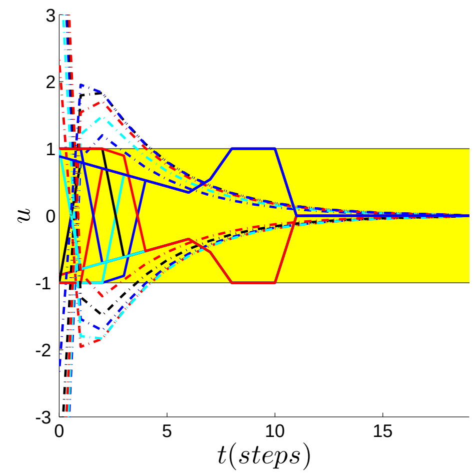


(b)

Figure 2. Nominal performance: ”boat-shaped” Lyapunov function of the switching function. The color of each point on the upper figure corresponds to the value of the same color on the lower figure.



(a) The representative state trajectories.



(b) The optimal control inputs.

Figure 3. Nominal performance: the closed-loop responses. Solid lines represent the closed-loop state trajectories using the proposed MP-SMC. Dashed-dotted lines represent the trajectories using the quasi-SMC from [11]. Various colors correspond to different initial conditions.

tion values

$$x(t+1) = \underbrace{\begin{bmatrix} 1 & 0.1 \\ 0.3 & 1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t) \tag{38}$$

$$\mathscr{X} : \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}}_{A_x} x(t) \leq \underbrace{\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}}_{b_x}, \quad \mathscr{U} : \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{A_u} u(t) \leq \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b_u}$$

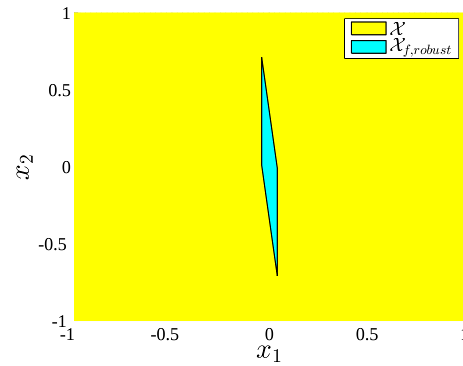
It is easy to see that the system is controllable. Choose $c_1 = 9$ so that one closed-loop eigenvalue is placed at 0.1. For the design of the MP-SMC, we set the prediction horizon N to 10, and ε to 0.1. We follow the design procedure discussed in the sliding mode condition to obtain the control invariant set $\mathscr{X}_f$ as shown in Fig. 1(a). With the boundary condition $\mathscr{X}_N = \mathscr{X}_f$, the j-th step feasible set is calculated backward for $j = N-1, \ldots, 0$. Fig. 1(b) shows that the j-th step feasible set keeps expanding, as j decreases, until the maximal control invariant set is reached. With the use of the multi-parametric programming toolbox (MPT) [19], the critical regions (defined as partitions of $\mathscr{X}_0$ where the optimal feedback law is continuous and affine [1]) are plotted in Fig. 1(c). The continuous convex polyhedral piecewise affine (PPWA) minimized objective function value is obtained as a function of the state $x \in \mathscr{X}_0$, which is shown in Fig. 2(a)-2(b). Its shape justifies its use as a Lyapunov function for proving the convergence of the switching function to 0.

The optimal control law is computed to be a piecewise affine function of the state $x \in \mathscr{X}_0$. To solve the optimization problem at each sampling step in real-time, we actually use the MPT Toolbox to generate a look-up table for the feedback control law in advance. Then we simulate the closed-loop system under the optimal control with several representative initial conditions. The closed-loop state trajectories are plotted in Fig. 3(a), and the corresponding inputs are plotted in Fig. 3(b). We use various colors to represent different initial conditions. The solid lines represent the proposed MP-SMC controller and the dashed-dotted lines represent the quasi-sliding mode controller in [11]. We can observe that the state and the input constraints are both satisfied using the proposed MP-SMC but not the quasi-SMC. The color of the state trajectory matches the color of the corresponding optimal control input. Some colors are repeatedly used, which could be easily distinguished and should not raise confusion.

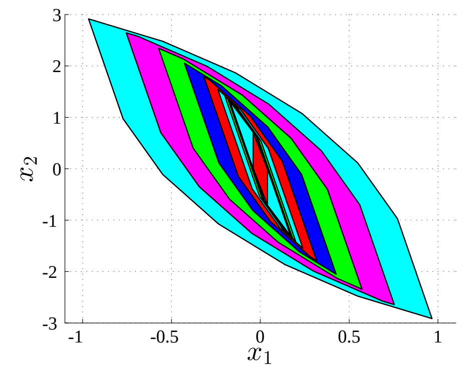
For robust performance validation, we consider the same system but with uncertainties, i.e.

$$\begin{aligned} x(t) &= Ax(t) + B(u(t) + d(t)) \\ -0.35 &\leq d(t) \leq 0.35 \end{aligned} \tag{39}$$

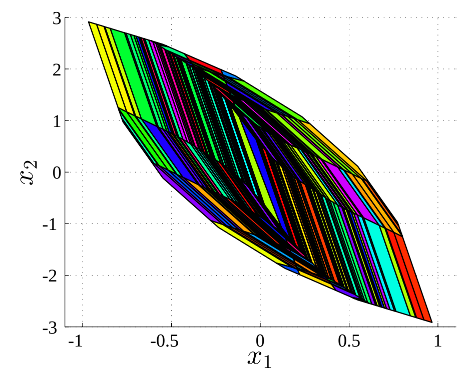
In simulation, the disturbance is uniformly distributed in that interval. For the CFTOC problem design, a new terminal constraint $\mathscr{X}_{f,robust}$ shown in Fig. 4(a) is designed to satisfy the robust sliding condition. After that, Eqn. (29) needs to be verified. Eqns. (30)-(33) are solved recursively using the MPT Toolbox. The j-th step robust feasible set and the critical regions are plotted in Fig. 4(b) and 4(c). The optimal objective function for the worst case is plotted in Fig. 5(a) and 5(b) versus the state. Its shape verifies that all feasible initial state is attracted to the sliding mode band regardless of disturbances. Some representative close-loop state trajectories (of different colors) are shown in Fig. 6(a). Similarly, solid lines represent the proposed MP-SMC and dashed-dotted lines represent the quasi-SMC from [11]. The corresponding robust MP-SMC inputs are in Fig. 6(b) while the quasi-SMC inputs are in Fig. 6(c). It has been shown that the system state is successfully regulated to the neighborhood of the origin in the presence of lumped model uncertainty and disturbance without violating the state and input constraints by the proposed MP-SMC but not the quasi-SMC.



(a) The design of $\mathscr{X}_{f,robust}$ enforcing the robust sliding condition.



(b) The j-th step robust feasible set (the set expands with decreasing j)



(c) The critical regions of the 0-th step robust feasible set

Figure 4. Robust performance: the feasible sets

CONCLUSION

An innovative model predictive sliding mode controller was proposed in this paper. The capability of dealing with system constraints and the robustness property has been inherited from MPC and SMC respectively. A constrained finite time optimal controller was designed to steer the state to the sliding manifold. The persistent feasibility and the closed-loop stability have been shown to be guaranteed if the (robust) control invariance condition is satisfied. The nominal and robust performance of the proposed controller was demonstrated by simulations.

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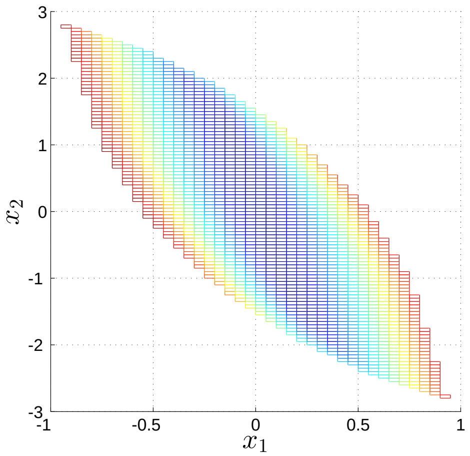
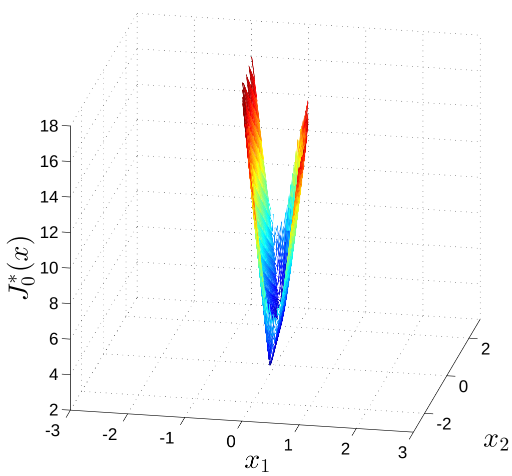


(a)



(b)

Figure 5. Robust performance: "boat-shaped" Lyapunov function of the switching function. The color of each point on the upper figure corresponds to the value of the same color on the lower figure.

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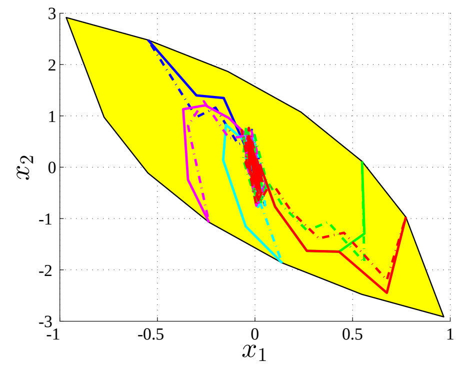
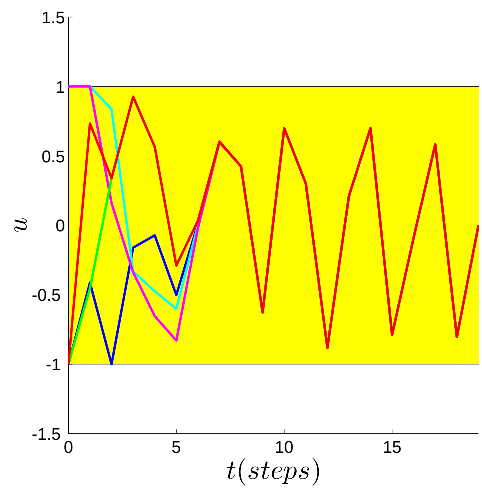
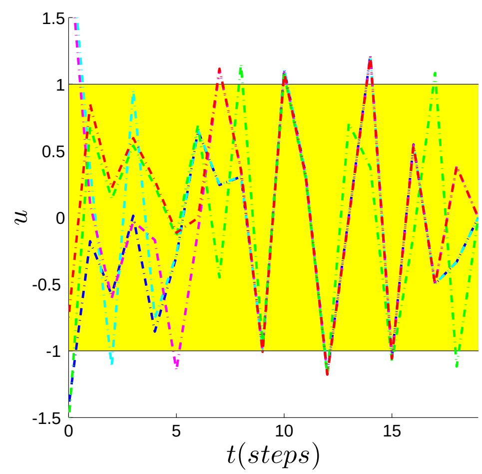


(a) The representative state trajectories.



(b) The MP-SMC optimal control inputs.



(c) The quasi-SMC control inputs.

Figure 6. Robust performance: the closed-loop responses. Solid lines represent the closed-loop state trajectories using the proposed MP-SMC. Dashed-dotted lines represent the trajectories using the quasi-SMC from [11]. Various colors correspond to different initial conditions.

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